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RESEARCH ARTICLE

Pricing cancellable American put options on the finite time horizon

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Abstract

The purpose of this paper is to present a numerical approach for pricing cancellable American put options, also known as game or Israeli options, on the finite time horizon. These options generalize the concept of American derivatives adding an early exercise right for the option's writer to the existing holder's right. The writer has to pay a penalty amount above the usual option payment to use this right. We first obtain the shape of the optimal regions for both participants. Then we approximate the optimal exercise boundaries maximizing the option's writer and holder financial expectations using some first exit properties of the Brownian motion. We also construct an efficient pricing algorithm based on these boundaries. A semiclosed form formula is derived when the underlying asset starts above the strike.

KEYWORDS

American cancellable options, exercise regions, game options, optimal boundaries, pricing

JEL CLASSIFICATION

C41, G12, G13

1 | INTRODUCTION

Option contracts are some of the most tradable derivative instruments in modern financial markets. They depend on an underlying asset and give to the option's holder the right to sell or buy it at a preagreed price, named strike price, until or at some maturity date. There are two main types of options—European and American. The basic difference between them is the moment they expire. The European contracts can be exercised only at the maturity date. Alternatively, an American option gives its holder the right to choose the expiration date. In such a way, he can capture immediately the payoff when the underlying asset reaches the desirable level. However, the option's holder is in a privileged position due to the essence of this early exercising right. Later, new financial instruments, called cancellable American options, appear to compensate for this imbalance. They give the option's writer the right to cancel the option prematurely paying some amount above the usual option payment. This way the writer's financial interest is protected from the possible large market deviations.

Two main questions arise for all American-style derivatives, particularly for the cancellable ones. First, is it optimal to exercise immediately at the actual underlying asset price? This question holds for both option participants when the American option exhibits a cancellable feature. The second important question is what is the fair option price if immediate exercise is not optimal, neither for the writer nor for the holder. We answer these questions in the present paper for finite maturity put options. The basic novelties of the current article can be summarized as follows. First, we examine options written on an asset that continuously pays dividends. It is well known that this is a realistic assumption and it leads to completely new model features. For example, if there are no dividends, then the cancellable American put turns to an American-style option which expires when the underlying asset hits the strike. This is a one-sided optimal stopping problem. On the contrary, early cancellation may be writer optimal not only at the strike in the presence of the dividends which leads to a two-sided problem. Second, we prove a series of propositions which determine the optimal regions for both options participants. On the basis of them, we construct a numerical algorithm to approximate the optimal boundaries. In such a way we know at every moment whether the exercise is profitable or not for the option's writer or holder. Also, we create a new relatively fast Monte Carlo algorithm to derive the fair option price when the asset starts between the optimal boundaries. Finally, we derive a semiclosed form formula for the option price when the initial asset price is above the strike.

Cancellable options are initially considered in Kifer (2000) in the framework of Dynkin games—see Dynkin (1967). Thus a two-sided optimal stopping problem arises because both participants strive to maximize their financial utilities. Usually these tasks are viewed as free boundary differential problems—we have to derive the solution as well as the region in which it holds—see, for example, Meyer (2016) and Guo et al. (2020). Alternatively, we shall attack directly the optimal stopping problem maximizing the financial results of the writer and holder.

It is well known that the option valuation problem is easier to solve when there are no maturity constraints, because the exercise boundaries are flat due to the absence of a coercive exercise at the maturity date. Emmerling (2012) and Zaevski (2020a) examine such call-style options, whereas Kyprianou (2004) and Suzuki and Sawaki (2007) obtain the corresponding results for the put options. Kunita and Seko (2004) and later Yam et al. (2014) explore the call versions when the maturity is finite. The put-style options under the assumption that the underlying asset does not pay dividends are considered in Kühn and Kyprianou (2007). In the present work, we remove this restriction. We establish our model using an extra discount factor. Together with the introduced time dependence in the payment structures, this additional discounting has another outstanding importance—a model based on a continuously paying dividend underlying asset can be expressed in this framework. The perpetual case is examined in Zaevski (2020b).

Our first task is to determine the shape of the optimal regions. It turns out that the holder's one for a fixed moment is an interval $(0, A)$ for some constant A less than the strike, $A < K$. The writer's region may be an interval $(B, K]$, $0 < A < B < K$, the singleton $\{K\}$, or the empty set. Once we know the form of the exercise regions, we use specific American-style derivatives to approximate the actual values of the optimal boundaries. These derivatives require one of the option's holders or writers to exercise when the asset reaches some boundary and gives the other the right to exercise earlier. In such a way these instruments allow one of the participants to maximize his financial result assuming that the other has a known strategy. A special role in our examination has a subclass of such derivatives. Their owner has the right to exercise at every moment before the maturity of receiving the usual option payment. In addition, if the underlying asset hits the strike when the remaining time to maturity is larger than some previously defined value τ, the derivative expires paying some amount η. We shall call this derivative a (τ, η)-American option. Obviously, if the time to maturity is less than τ, then the (τ, η)-American option coincides with the ordinary American one.

Our results, summarized w.r.t. the time to maturity, are as follows. The holder's exercise boundary starts from a point less than or equal to the strike and decreases to the perpetual value. There are three possibilities for the writer's boundary. In all of them it does not exist for small enough maturities—it is more profitable for the writer to wait for maturity instead of cancelling and paying the penalty. This is always true for some very large penalties—thus the option turns to noncancellable. Otherwise, small enough penalties lead to the strike for the writer's boundary for some medium maturities. It may stay always equal to the strike after the initial period turning the option to a (τ, η)-American put. Alternatively, it may fall below the strike in some moment after which it decreases to the perpetual value.

Once the exercise boundaries are derived, the cancellable option pricing problem turns to a partial differential equation in a known region. Its solution can be expressed as the expectation related to the first hitting moment of the asset to these boundaries. The continuation region consists of two parts—one between the optimal boundaries and one above the strike. Hence, we have to examine them separately. If the asset starts between the boundaries, then we have a two-sided exit problem. We create a relatively fast and efficient Monte Carlo algorithm to derive the fair option price. Otherwise, if the initial asset price is above the strike we have a one-sided hitting problem. In this case we derive a semiclosed form formula which depends on the prices of some ordinary American options. Their maturity is the critical time value at which the writer's boundary appears.

Also we provide various numerical experiments to validate the derived theoretical results. That way we confirm their applicability and economical importance. The constructed algorithms are applied to different values of the parameters which describe all possible cases for the option features. Thus the accuracy and efficiency of the proposed approach are checked. We summarize the produced numerical results in several figures and tables. The impact of the writer's early cancelling right is evaluated through a comparison with the corresponding noncancellable American option.

The paper is organized as follows. In Section 2 we introduce the model and give the base we use later. In Section 3 we obtain the shape of the exercise regions. In Section 4 we motivate our approach for deriving the optimal boundaries. The option pricing problem is considered in Section 5. Some numerical examples are conferred in Section 6. We conclude in Section 7.

2 | PRELIMINARIES

Let B_t be a Brownian motion under the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, Q)$. We assume that the measure Q is risk-neutral and the underlying asset follows a log-normal process w.r.t. it,

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma B_t\right). \tag{1}$$

Let the risk-free rate be the constant r and the discount rate be the positive constant $\lambda > 0$. We impose that the total discount rate is positive too, $r + \lambda > 0$. Note that negative values for the risk-free rate are admissible. Let $T < \infty$ be the maturity date and $\mathcal{T}_{[t,T]}$ and $\overline{\mathcal{T}}_{[t,T]}$ be the sets of all stopping times with values in $[t, T]$ and $[t, T]\bigcup\{\infty\}$, respectively. They will be associated with the holder's and writer's strategies. The state $\{\infty\}$ means that the writer will not cancel earlier. Note that the elements of the set $\mathcal{T}_{[t,T]}$ are always finite since the option must be exercised, even this is done on the expiration date T. Let $\eta > 0$ be the penalty. In such a way the payment structure for a cancellable put option is

$$\begin{aligned} N_1(t,x) &= e^{-\lambda t}(K - x)^+, \\ N_2(t,x) &= e^{-\lambda t}(\eta + (K - x)^+), \end{aligned} \tag{2}$$

that is, the writer owes an amount of $N_2(t,x)$ if he cancels the option in moment t at price $S_t = x$. Analogously, he is obliged to pay $N_1(t,x)$ if the option's holder exercises.

Let us discuss how a dividend-paying model can be expressed in these terms. If we denote the dividend rate by δ and the corresponding model by the (r, λ, δ)-model, then it is equivalent to the $(r - \delta, \lambda + \delta, 0)$-model. The proof of this relation is based on the fact that after discounting with rate $r - \delta$ the underlying asset has to be a martingale under the (r, λ, δ)-model—for more details see proposition 2.3 from Zaevski (2020a). All this allows us to work with the $(r, \lambda, 0)$-model without loss of generality.

Next we establish the optimal strategies for both options participants.

Definition 2.1. The writer's and holder's optimal strategies are defined as follows.

1. Let ζ be a stopping time from the set $\mathcal{T}_{[t,T]}$. Then the ζ-writer's optimal strategy $\phi^s(\zeta; x) \in \overline{\mathcal{T}}_{[t,T]}$ is the stopping time which minimizes

$$E^{t,x}\begin{bmatrix} e^{-r(\zeta - t)} N_1(\zeta, S_\zeta) I_{\zeta \le \phi^s(\zeta;\cdot)} \\ + e^{-r(\phi^s(\zeta;\cdot) - t)} N_2(\phi^s(\zeta;\cdot), S_{\phi^s(\zeta;\cdot)}) I_{\phi^s(\zeta;\cdot) < \zeta} \end{bmatrix}. \tag{3}$$

2. Analogously, the ζ-holder's optimal strategy, $\phi^b(\zeta; x) \in \mathcal{T}_{[t,T]}$, is the stopping time which maximizes

$$E^{t,x}\begin{bmatrix} e^{-r(\phi^b(\zeta;\cdot) - t)} N_1(\phi^b(\zeta;\cdot), S_{\phi^b(\zeta;\cdot)}) I_{\phi^b(\zeta;\cdot) \le \zeta} \\ + e^{-r(\zeta - t)} N_2(\zeta, S_\zeta) I_{\zeta < \phi^b(\zeta;\cdot)} \end{bmatrix}. \tag{4}$$

for $\zeta \in \overline{\mathcal{T}}_{[t,T]}$.

Next we define the optimal regions.

Definition 2.2. The holder's and writer's exercise regions for a moment $t < T$, $\Upsilon^b(t)$ and $\Upsilon^s(t)$, and the continuation region $\overline{\Upsilon}(t)$ are defined as follows.

1. The point $(t, x) \in \Upsilon^b(t)$ if for every stopping time $\zeta \in \mathcal{T}_{[t,T]}$,

$$N_1(t, x) \geq E^{t,x}\begin{bmatrix} e^{-r(\zeta-t)}N_1(\zeta, S_\zeta)I_{\zeta \leq \phi^s(\zeta;\cdot)} \\ + e^{-r(\phi^s(\zeta;\cdot)-t)}N_2(\phi^s(\zeta;\cdot), S_{\phi^s(\zeta;\cdot)})I_{\phi^s(\zeta;\cdot)<\zeta} \end{bmatrix}. \tag{5}$$

2. The point $(t, x) \in \Upsilon^s(t)$ if for every stopping time $\zeta \in \overline{\mathcal{T}}_{[t,T]}$,

$$N_2(t, x) \leq E^{t,x}\begin{bmatrix} e^{-r(\phi^b(\zeta;\cdot)-t)}N_1(\phi^b(\zeta;\cdot), S_{\phi^b(\zeta;\cdot)})I_{\phi^b(\zeta;\cdot)\leq\zeta} \\ + e^{-r(\zeta-t)}N_2(\zeta, S_\zeta)I_{\zeta<\phi^b(\zeta;\cdot)} \end{bmatrix}. \tag{6}$$

3. The continuation region at the moment t is $\overline{\Upsilon}(t) = \mathbb{R}_+ \backslash (\Upsilon^b(t) \cup \Upsilon^s(t))$.

4. The corresponding total sets Υ^b, Υ^s, and $\overline{\Upsilon}$ are defined as $\Upsilon^b = \{(t, x) | x \in \Upsilon^b(t)\}$, $\Upsilon^s = \{(t, x) | x \in \Upsilon^s(t)\}$, and $\overline{\Upsilon} = \{(t, x) | x \in \overline{\Upsilon}(t)\}$, respectively.

We shall prove now a proposition which gives the time dependence in option pricing.

Proposition 2.1. *Let the maturity date be T. If the price of a cancellable option at moment t is $V(t, T, S_t)$ for some function $V(t, T, x)$, then*

$$V(t, T, x) = e^{-\lambda t}V(0, T - t, x). \tag{7}$$

Proof. Let ζ_1 and ζ_2 be the first moments at which the underlying asset reaches the optimal regions Υ^b and Υ^s, respectively. Using the Markovian property we conclude

$$\begin{aligned} V(t, T, x) &= E^{t,x}\left[e^{-r((\zeta_1\wedge T)-t)}N_1(\zeta_1 \wedge T, S_{\zeta_1\wedge T})I_{\zeta_1\wedge T\leq\zeta_2}\right] \\ &\quad + E^{t,x}\left[e^{-r(\zeta_2-t)}N_2(\zeta_2, S_{\zeta_2})I_{\zeta_2<\zeta_1\wedge T}\right] \\ &= e^{-\lambda t}E^{t,x}\left[e^{-r((\zeta_1\wedge T)-t)}e^{-\lambda((\zeta_1\wedge T)-t)}N_1(0, S_{\zeta_1\wedge T})I_{\zeta_1\wedge T\leq\zeta_2}\right] \\ &\quad + e^{-\lambda t}E^{t,x}\left[e^{-r(\zeta_2-t)}e^{-\lambda(\zeta_2-t)}N_2(0, S_{\zeta_2})I_{\zeta_2<\zeta_1\wedge T}\right] \\ &= e^{-\lambda t}V(0, T - t, x). \end{aligned} \tag{8}$$

□

The way we determine the exercise regions in Definition 2.2 together with the form of payment structure (2) leads to another proposition related to the time dependence.

Proposition 2.2. *Let us denote by $\alpha(t, T)$ and $\beta(t, T)$ the holder's and writer's exercise boundaries at the moment t if the maturity is T. Let also $a(\tau)$ and $b(\tau)$ be the corresponding boundaries at the initial moment provided that the time to maturity is τ. Then $\alpha(t, T) = a(T - t)$ and $\beta(t, T) = b(T - t)$.*

Propositions 2.1 and 2.2 have a major importance. First, the future option price can be computed using the initial price function and the time to maturity. Also, the exercise boundaries depend only on the remaining time to maturity, not on the concrete values of t and T. These facts allow us to use an alternative parametrization together with the presented above one. Namely, we shall use the time to maturity, denoted by τ, instead of the maturity date assuming that the initial moment is $t = 0$. The sets in Definition 2.2 will be denoted by $\mathcal{T}_\tau$, $\overline{\mathcal{T}}_\tau$, Υ^b_τ, Υ^s_τ, and $\overline{\Upsilon}_\tau$ instead $\mathcal{T}_{[t,T]}$, $\overline{\mathcal{T}}_{[t,T]}$, $\Upsilon^b(t)$, $\Upsilon^s(t)$, and $\overline{\Upsilon}(t)$, respectively. We can see that the relation between them is $\mathcal{T}_{[t,T]} \equiv \mathcal{T}_{T-t}$, $\overline{\mathcal{T}}_{[t,T]} \equiv \overline{\mathcal{T}}_{T-t}$, $\Upsilon^b(t) \equiv \Upsilon^b_{T-t}$, $\Upsilon^s(t) \equiv \Upsilon^s_{T-t}$, and $\overline{\Upsilon}(t) \equiv \overline{\Upsilon}_{T-t}$.

3 | EXERCISE REGIONS

If we read carefully the proofs of propositions 2.2–2.4 from Zaevski (2020b), related to the perpetual options, we can see that they are true when the maturity is finite too. We summarize them in the following proposition.

Proposition 3.1. *The following three statements hold.*

1. *If $x > K$, then $x \in \overline{\Upsilon}_\tau$ for an arbitrary τ.*
2. *If $x \in \Upsilon^b_\tau$ and $y < x$, then $y \in \Upsilon^b_\tau$ too.*
3. *If $x \in \Upsilon^s_\tau$ (note that $x \le K$) and $x \le y \le K$, then $y \in \Upsilon^s_\tau$ too.*

This proposition shows that for a fixed moment t the holder's exercise region has the form $\Upsilon^b(t) = (0, a)$ for some constant a. On the contrary, we can expect that the writer's region is the empty set, the singleton $\{K\}$, or an interval with the strike for the upper boundary.

Now we need the following lemma which is very similar to one presented in Kunita and Seko (2004), lemma 3.3.

Lemma 3.1. *The price function of a cancellable option is nondecreasing w.r.t. the time to maturity.*

Using Lemma 3.1 we shall prove a proposition which provides the behavior of the holder's and writer's optimal boundaries—$a(\tau)$ and $b(\tau)$, respectively.

Proposition 3.2. *Both boundaries are nonincreasing functions w.r.t. the time to maturity.*

Proof. We denote again the option price by $V(\tau, x)$ where τ is the time to maturity and x is the initial asset price. Let ϵ be arbitrary and δ be small enough. Using Lemma 3.1 and $a(\tau) + \delta \notin \Upsilon^b_\tau$ we conclude

$$\begin{aligned} V(\tau + \varepsilon, a(\tau) + \delta) &\ge V(\tau, a(\tau) + \delta) \\ &> e^{-\lambda\tau}(K - (a(\tau) + \delta))^+ \\ &\ge e^{-\lambda(\tau+\varepsilon)}(K - (a(\tau) + \delta))^+. \end{aligned} \tag{9}$$

Therefore $a(\tau) + \delta \notin \Upsilon^b_{\tau+\varepsilon}$ too. Hence, $a(\tau) + \delta > a(\tau + \varepsilon)$. Taking $\delta \to 0$ we derive $a(\tau) \ge a(\tau + \varepsilon)$. Analogously, we use that $b(\tau) - \delta \notin \Upsilon^s_\tau$ to obtain

$$\begin{aligned} V(\tau - \varepsilon, b(\tau) - \delta) &\le V(\tau, b(\tau) - \delta) \\ &< e^{-\lambda\tau}((K - (b(\tau) - \delta))^+ + \eta) \\ &\le e^{-\lambda(\tau-\varepsilon)}((K - (b(\tau) - \delta))^+ + \eta). \end{aligned} \tag{10}$$

Hence, $b(\tau) - \delta \notin \Upsilon^s_{\tau-\varepsilon}$ and therefore $b(\tau) - \delta < b(\tau - \varepsilon)$, which leads to $b(\tau) \le b(\tau - \varepsilon)$. □

We shall examine now the option's behavior near the maturity. Let us denote by $V_{\mathrm{am}}(\tau, x)$ the price of an American option with the same parameters, but without the writer's cancelling right.

Proposition 3.3. *Let τ_1 be the largest value (possibly infinity) for the time to maturity, below which the price of the American option $V_{\mathrm{am}}(\tau, K)$ is less than the penalty η. Note that τ_1 exists since $V_{\mathrm{am}}(\tau, K) \to 0$ for $\tau \to 0$. Then the writer's optimal region Υ^s_τ is empty for every $\tau \le \tau_1$ and vice versa—the set Υ^s_τ is not empty for $\tau > \tau_1$.*

Also, let us denote by $\overline{\eta}$

$$\bar{\eta} = K\frac{q^q}{(q+1)^{q+1}} \tag{11}$$

for

$$q = \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}} + \left(\frac{r}{\sigma^2} - \frac{1}{2}\right). \tag{12}$$

We have that $\tau_1 < \infty$ *if* $\eta < \bar{\eta}$ *and* $\tau_1 = \infty$, *otherwise.*

Proof. Theorem 6.2 of Zaevski (2021b) gives that the price of the perpetual American put option $V_{\mathrm{am}}(\infty, K)$ is given just by formula (11). Hence, if $\eta \geq \bar{\eta}$, then $V_{\mathrm{am}}(\tau, K) < \eta$ for all $\tau > 0$.[1] Therefore $\tau_1 = \infty$. In the opposite case $\eta < \bar{\eta}$, we have $\tau_1 < \infty$.

Suppose that there exists $\tau \leq \tau_1$, such that $\Upsilon_\tau^s \neq \emptyset$. Hence, there exists some $x \leq K$, such that $x \in \Upsilon_\tau^s$. Using the third statement of Proposition 3.1 we see that $K \in \Upsilon_\tau^s$ too. Suppose that the writer does not cancel earlier. Hence, the option turns to an ordinary American put. We have assumed above that its price is less than the penalty and therefore the strategy to do nothing gives a better result for the writer than the immediate exercise. Therefore the set Υ_τ^s is empty for all $\tau \leq \tau_1$.

Suppose now that $\Upsilon_\tau^s = \emptyset$ for some $\tau > \tau_1$. Note that this is true for all $t < \tau$ due to the third statement of Propositions 3.1 and 3.2. Hence the option is ordinary American. This means that never cancelling has to be the optimal strategy for the writer of an at-the-money option. It leads to the financial result $V_{\mathrm{am}}(\tau_1, K)$, which however is more than η due to $\tau > \tau_1$ and therefore the immediate exercise would be writer preferable. This observation finishes the proof. □

We shall denote hereafter the price of (τ_1, η)-American put option by $V_{(\tau_1,\eta)}(\tau, x)$. Let $\tau_2 \geq \tau_1$ be this value of the time to maturity, possibly infinity, at which the writer's exercise boundary detaches from the strike. Note that Proposition 3.2 shows that once the writer's boundary falls below the strike, it never returns back. The following proposition stands.

Proposition 3.4. *Let* $\tau > \tau_1$. *The writer's exercise region consists only of the strike,* $\Upsilon_\tau^s = \{K\}$, *if* $\frac{d}{dx}V_{(\tau_1,\eta)}(\tau, K^-) \geq -1$. *If* Υ_τ^s *is an interval, then* $\frac{d}{dx}V_{(\tau_1,\eta)}(\tau, K^-) \leq -1$.

Something more, if the writer's optimal boundary for a perpetual cancellable put option with the same parameters is equal to the strike, then $\tau_2 = \infty$. *Otherwise,* $\tau_2 < \infty$.

Proof. If the writer's exercise region is the singleton $\{K\}$, then $(x, K) \in \overline{\Upsilon}_\tau$ for some $x < K$. Therefore, $V_{(\tau_1,\eta)}(\tau, y) < N_2(0, y)$ for all $y \in (x, K)$. Taking the limit $y \to K$ we derive

$$\begin{aligned} \frac{d}{dx}V_{(\tau_1,\eta)}(\tau, K^-) &= \lim_{y \to K^-} \frac{V_{(\tau_1,\eta)}(\tau, y) - V_{(\tau_1,\eta)}(\tau, K)}{y - K} \\ &\geq \lim_{y \to K^-} \frac{N_2(0, y) - N_2(0, K)}{y - K} \\ &= \frac{\partial N_2(0, K^-)}{\partial x} = -1. \end{aligned} \tag{13}$$

Otherwise, suppose that the writer's boundary is below the strike for some τ. Therefore there exists some $x < K$ such that $V(\tau, y) = N_2(0, y)$ for all $y \in (x, K)$. Note that $V_{(\tau_1,\eta)}(\tau, y) \geq V(\tau, y)$. Using similar arguments as in inequality (13) we derive $\frac{d}{dx}V_{(\tau_1,\eta)}(\tau, K^-) \leq -1$.

We finish the proof by the observation that if the writer's exercise region of the perpetual option is the singleton $\{K\}$, then $\tau_2 = \infty$ and vice versa. □

[1]Note that the price of an American option is nondecreasing w.r.t. the time to maturity.

Remark 3.1. Note that the value of $\frac{d}{dx}V_{(\tau_1,\eta)}(\tau_1, K^-)$ is larger than -1 due to the smooth fit principle for the noncancellable American options and the convexity of their price functions. Thus τ_2 is featured as the moment at which $\frac{d}{dx}V_{(\tau_1,\eta)}(\tau, K^-)$ falls below -1.

If we use the notations with the current moment and the maturity, t and T, instead of the time to maturity τ, the moment τ_2 is characterized as the lowest moment, if it exists, at which

$$\frac{d}{dx}V_{(\tau_1,\eta)}(t, T, K^-) > -e^{-\lambda t}. \tag{14}$$

In the following theorem we summarize the derived results for the form of the exercise regions of a cancellable put option.

Theorem 3.1. *The holder's exercise boundary is a decreasing function starting from the point*

$$\min\left(\frac{r+\lambda}{\lambda}, 1\right)K. \tag{15}$$

Note that Equation (15) *gives the optimal boundary of an American option at the maturity—see, for example, proposition* 3.5 *from Zaevski* (2021*b*).

The form of the writer's exercise boundary is more complicated. Let B be its perpetual value, if it exists.[2] *Hence*

1. *If B does not exist, equivalently to* $\eta \geq \bar{\eta}$ *for* $\bar{\eta}$ *as in Equation* (11), *then* $\tau_1 = \tau_2 = \infty$ *and* $\Upsilon^s \equiv \varnothing$.
2. *If* $B = K$, *then* $\tau_1 < \infty$, *but* $\tau_2 = \infty$. *Therefore* $\Upsilon^s_\tau \equiv \varnothing$ *for* $\tau \leq \tau_1$ *and* $\Upsilon^s_\tau \equiv \{K\}$ *otherwise.*[3]
3. *If* $B < K$, *then* $\tau_1 < \tau_2 < \infty$. *Thus the writer's exercise boundary does not exist for* τ *less than* τ_1, *it coincides with the strike for* $\tau \in (\tau_1, \tau_2)$, *and it is a decreasing tending to B function for* $\tau \geq \tau_2$.

In such a way the option is American when $\tau \in (0, \tau_1]$, *it is* (τ_1, η)*-American for* $\tau \in (\tau_1, \tau_2)$, *and a real cancellable option for* $\tau \in [\tau_2, \infty)$.

4 | DERIVING THE EXERCISE BOUNDARIES

Let us first define the following European-style derivatives for some functions $0 < a(t) < b(t)$. They expire at the maturity date or when the underlying asset exits the strip $(a(t), b(t))$. The derivatives pay an amount of $N_1(t, a(t))$ or $N_2(t, b(t))$ if the exit happens from the lower or upper boundary, respectively. We shall name these derivatives $(a(t), b(t))$-European options.

We shall construct now an approximation algorithm for both exercise boundaries. Proposition 2.1 allows us to derive their values in some future moment t as the values at the initial moment of an option with a lower maturity $T - t$. Assume that the time to maturity is large enough. Note that the moment τ_1 can be obtained numerically—Proposition 3.3 says that the price of an at-the-money noncancellable American option maturating after τ_1 time is equal to the penalty. Unfortunately, this is not the case for the second important moment τ_2. Proposition 3.4 states that τ_2 is characterized by the derivative of the price of a (τ_1, η)-American option. Usually, such instruments can be priced only numerically with some precision. This does not allow the limit in the derivative to be calculated with a sufficient accuracy. Hence, we shall approximate τ_2 as the largest value of τ for which our algorithm returns the strike for the writer's optimal boundary.

The first thing we need to do is to obtain which case of Theorem 3.1 is actual. If the penalty is larger than the critical value $\bar{\eta}$, given in Equation (11), then the option is an ordinary American put. Suppose now that $\eta < \bar{\eta}$. We have to

[2]It can be obtained by the use of theorem 3.1 of Zaevski (2020b).
[3]Note that this is the actual case when $r \geq 0$ due to proposition 2.5 from Zaevski (2020b).

calculate the writer's exercise boundary of the perpetual cancellable put option, B, using the presented in Zaevski's (2020b) approach. It can be equal or less than the strike. We shall examine separately both cases later.

The next step is to divide the time to maturity interval into $n \geq 2$-subintervals, $0 \equiv t_0 < t_1 < \cdots < t_n \equiv T \equiv \tau$. We can think that $\tau_1 < \tau$, because in the opposite case the option is noncancellable. We impose two requirements—first, τ_1 to be a grid node and, second, the division to be relatively uniform. To do this we use the following procedure. First, we divide the interval into two parts—$(0, \tau_1)$ and (τ_1, τ). After that we divide uniformly both intervals into p and q parts, respectively, such that $p + q = n$ and

$$p = \min\left(\max\left(1, Round\left(\frac{\tau_1}{\tau}n\right)\right), n - 1\right). \tag{16}$$

We use above the notation $Round(x)$ for the nearest to x integer. Formulation (16) guarantees that $p \geq 1$ and $q \geq 1$, that is, there is at least one subinterval before τ_1 as well as after τ_1. Also, it is important to note that $t_p = \tau_1$.

We shall approximate the holder's and writer's exercise boundaries by exponents of piecewise linear functions

$$\begin{aligned} a(t) &= \sum_{i=1}^{n}\exp(a_i(t))I_{t\in(t_{i-1},t_i]} \equiv \sum_{i=1}^{n}\exp(a_{1,i}t + a_{2,i})I_{t\in(t_{i-1},t_i]}, \\ b(t) &= \sum_{i=1}^{p}\exp(b_i(t))I_{t\in(t_{i-1},t_i]} \equiv \sum_{i=1}^{p}\exp(b_{1,i}t + b_{2,i})I_{t\in(t_{i-1},t_i]}, \end{aligned} \tag{17}$$

respectively. We assume that $a(t) < b(t) \leq K$ for every t less the maturity. We require continuity at the grid nodes —$a_i(t_i) = a_{i-1}(t_i)$ and $b_i(t_i) = b_{i-1}(t_i)$—we shall denote by $A_0, A_1, \ldots, A_n$ and $B_0, B_1, \ldots, B_p$ the corresponding values. Note that we have proved in Theorem 3.1 that it is never optimal for the writer to cancel if $\tau \leq \tau_1$ and therefore the writer's exercise boundary does not exist for $i > p$. Let

$$G(x, T; a(t), b(t)) \equiv G(x, \tau; \{t_0, \ldots, t_n\}, \{A_0, \ldots, A_n\}, \{B_0, \ldots, B_p\}) \tag{18}$$

be the price of the $(a(t), b(t))$-European option if the initial asset value is x and the time to maturity is $\tau \equiv T$. We assume $x \in (A_0, B_0)$. Let us denote by ζ_1 and ζ_2 the first hitting times of the underlying asset to the functions $a(t)$ and $b(t)$, respectively, and let $\zeta = \zeta_1 \wedge \zeta_2$. Since the asset price is described by the log-normal process (1), the stopping times ζ_1 and ζ_2 can be viewed as the first hitting times of the Brownian motion to the piecewise linear boundaries

$$\begin{aligned} c(t) &= \sum_{i=1}^{n} c_i(t)I_{t\in(t_{i-1},t_i]} \equiv \sum_{i=1}^{n}(c_{1,i}t + c_{2,i})I_{t\in(t_{i-1},t_i]}, \\ d(t) &= \sum_{i=1}^{p} d_i(t)I_{t\in(t_{i-1},t_i]} \equiv \sum_{i=1}^{p}(d_{1,i}t + d_{2,i})I_{t\in(t_{i-1},t_i]} \end{aligned} \tag{19}$$

for

$$\begin{aligned} c_{1,i} &= \frac{a_{1,i} - r}{\sigma} + \frac{\sigma}{2}, \quad i = 1, \ldots, n, \\ c_{2,i} &= \frac{a_{2,i} - \ln(x)}{\sigma}, \quad i = 1, \ldots, n, \\ d_{1,i} &= \frac{b_{1,i} - r}{\sigma} + \frac{\sigma}{2}, \quad i = 1, \ldots, p, \\ d_{2,i} &= \frac{b_{2,i} - \ln(x)}{\sigma}, \quad i = 1, \ldots, p. \end{aligned} \tag{20}$$

Thus the $(a(t), b(t))$-European option price turns to

$$\begin{aligned}
G(x,T;a(t),b(t)) &= E^x\left[e^{-(r+\lambda)(\zeta_1\wedge T)}\left(K-S_{\zeta_1\wedge T}\right)^+ I_{(\zeta_1\wedge T)\le\zeta_2}\right] \\
&\quad + E^x\left[e^{-(r+\lambda)\zeta_2}((K-S_{\zeta_2})^+ + \eta) I_{\zeta_2<(\zeta_1\wedge T)}\right] \\
&= K\sum_{i=1}^{n} E\left[e^{-(r+\lambda)\zeta_1} I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta_1}\right] - x\sum_{i=1}^{n} e^{\sigma c_{2,i}} E\left[e^{-\psi_{1,i}\zeta_1} I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta_1}\right] \\
&\quad + (K+\eta)\sum_{i=1}^{p} E\left[e^{-(r+\lambda)\zeta_2} I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta_2}\right] - x\sum_{i=1}^{p} e^{\sigma d_{2,i}} E\left[e^{-\psi_{2,i}\zeta_2} I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta_2}\right] \\
&\quad + Ke^{-(r+\lambda)T} Q(B_T<k, T\le\zeta) - xe^{-\psi_3 T} E\left[e^{\sigma B_T} I_{B_T<k, T\le\zeta}\right],
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
\psi_{1,i} &= (r+\lambda) - \left(r-\frac{\sigma^2}{2}\right) - \sigma c_{1,i} = \lambda + \frac{\sigma^2}{2} - \sigma c_{1,i}, \\
\psi_{2,i} &= (r+\lambda) - \left(r-\frac{\sigma^2}{2}\right) - \sigma d_{1,i} = \lambda + \frac{\sigma^2}{2} - \sigma d_{1,i}, \\
\psi_3 &= \lambda + \frac{\sigma^2}{2}, \\
k &= \frac{1}{\sigma}\ln\left(\frac{K}{x}\right) - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)T.
\end{aligned} \tag{22}$$

We use formulas (A13) from Proposition A.7 to calculate the expectations in the first, second, third, and forth terms of formula (21). The fifth and sixth terms can be obtained from Equation (A14) of Proposition A.8. We use Laplace transform (A14) taken in the point zero for the probability in the fifth term.

We shall work backwards to derive the values of $A_0, A_1, ..., A_n$ and $B_0, B_1, ..., B_n$. Formula (15) leads to

$$A_n = \min\left(\frac{r+\lambda}{\lambda}, 1\right)K. \tag{23}$$

Since the option is noncancellable for $\tau < \tau_1$, we derive the values $A_p, A_{p+1}, ..., A_n$ as in Zaevski (2021b). Also, the writer's exercise boundary in the critical point τ_1 is equal to the strike and therefore $B_p = K$. Now we have to examine separately the cases when the writer's exercise boundary is always the strike or not.

4.1 | Writer's exercise boundary equal to the strike

Suppose that the writer's exercise boundary of the perpetual option is equal to the strike. Hence, this is true for the finite maturity case too and therefore $B_1 = B_2 = \cdots = B_p = K$. Thus the problem for pricing a cancellable option turns to a pricing problem for a (τ_1, η)-American put. The price function (21) turns to

$$\begin{aligned}
G(x,T;a(t),b(t)\equiv K) &= E^x\left[e^{-(r+\lambda)(\zeta_1\wedge T)}\left(K-S_{\zeta_1\wedge T}\right)^+ I_{(\zeta_1\wedge T)\le\zeta_2}\right] \\
&\quad + \eta E^x\left[e^{-(r+\lambda)\zeta_2} I_{\zeta_2<(\zeta_1\wedge T)}\right] \\
&= K\sum_{i=1}^{n} E\left[e^{-(r+\lambda)\zeta_1} I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta_1}\right] - x\sum_{i=1}^{n} e^{\sigma c_{2,i}} E\left[e^{-\psi_{1,i}\zeta_1} I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta_1}\right] \\
&\quad + \eta\sum_{i=1}^{p} E\left[e^{-(r+\lambda)\zeta_2} I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta_2}\right] \\
&\quad + Ke^{-(r+\lambda)T} Q(B_T<k, T\le\zeta) - xe^{-\psi_3 T} E\left[e^{\sigma B_T} I_{B_T<k, T\le\zeta}\right].
\end{aligned} \tag{24}$$

We have to derive the values of the holder's optimal boundary. Suppose that we know the values $A_m, A_{m+1}, ..., A_n$ for some $m \le p$. We want to find the value of A_{m-1} which maximizes the holder's financial result. Let us examine a

(τ_1, η)-American option with time to maturity $T - t_{m-1}$. Suppose that the initial asset value is fixed to x. We shall find the value for $A_{m-1} \leq x$, for which the price of an $(a(t + t_{m-1}), b(t + t_{m-1}))$-European option with maturity $T - t_{m-1}$,

$$G(x, T - t_{m-1}; \{0, t_m - t_{m-1}, \ldots, t_n - t_{m-1}\}\{A_{m-1}, \ldots, A_n\}, \{K, \ldots, K\}), \tag{25}$$

is largest. If the true holder's exercise boundary is indeed an exponent of a piecewise linear function, then the value A_{m-1} should be one and the same for all $x \in \overline{\Upsilon}_{T-t_{m-1}}$. But we cannot expect such a thing and therefore we have to work in a different way. Let us denote by $\alpha(x)$ the negative number, which maximizes the following $(a(t + t_{m-1}), b(t + t_{m-1}))$-European option price:

$$G(x, T - t_{m-1}; \{0, t_m - t_{m-1}, \ldots, t_n - t_{m-1}\}\{e^{\alpha}x, \ldots, A_n\}, \{K, \ldots, K\}). \tag{26}$$

We search for the largest value of x, for which $\alpha(x) = 0$. In the terms of the American derivatives it approximates the holder's exercise boundary, because it is the largest value of the initial asset price for which immediate exercising would be optimal. Hence, namely, this value of x is our approximation of A_{m-1}. Although Figure 1a does not present the actual case, it is informative how we derive the holder's exercise boundary. A red point is marked the biggest x such that $\alpha(x) = 0$ and just it is our approximation of the boundary.

4.2 | Writer's exercise boundary below the strike

Assume now that the writer's exercise boundary of the corresponding perpetual option is less than the strike. Therefore the second important moment at which the boundary detaches from the strike is finite, $\tau_2 < \infty$. As we mentioned above we cannot derive this moment via Proposition 3.4. We shall build a similar but two-sided algorithm to derive the writer's exercise boundary as well as the holder's one.

Suppose that we have derived the values $A_m, A_{m+1}, \ldots, A_n$ and $B_m, \ldots, B_p$ for some $m \leq p$. We shall derive first the value B_{m-1}. Let us fix some x below the strike and some β in the interval $[0, \ln K - \ln x]$. For a fixed positive β we denote by $\alpha(x, \beta)$ the negative value of α which maximizes the price of the following $(a(t + t_{m-1}), b(t + t_{m-1}))$-European option with time to maturity $T - t_{m-1}$:

$$G(x, T - t_{m-1}; \{0, t_m - t_{m-1}, \ldots, t_n - t_{m-1}\}\{e^{\alpha}x, \ldots, A_n\}, \{e^{\beta}x, B_m, \ldots, B_p\}). \tag{27}$$

Let $\beta(x)$ be the value of β which minimizes

$$G(x, T - t_{m-1}; \{0, t_m - t_{m-1}, \ldots, t_n - t_{m-1}\}\{e^{\alpha(x,\beta)}x, \ldots, A_n\}, \{e^{\beta}x, B_m, \ldots, B_p\}). \tag{28}$$

We search for the lowest x for which $\beta(x) = 0$. Namely, this is our approximation for the writer's boundary at the moment t_{m-1}. In fact this x is the lowest value for the underlying asset for which immediate cancelling is optimal for the writer.

We derive analogously the holder's boundary value A_{m-1}. Let for a fixed x, $0 < x < K$, and $\alpha \in (-\infty, 0]$, $\beta(x, \alpha)$ minimizes price (27) in the interval $[0, \ln K - \ln x]$. Let also $\alpha(x)$ be the value which maximizes

$$G(x, T - t_{m-1}; \{0, t_m - t_{m-1}, \ldots, t_n - t_{m-1}\}\{e^{\alpha}x, \ldots, A_n\}, \{e^{\beta(x,\alpha)}x, B_m, \ldots, B_p\}). \tag{29}$$

We search for the highest x for which $\alpha(x) = 0$. As we mentioned above, this is the highest value for the underlying asset which makes the immediate exercise optimal for the option's holder.

In Figure 1 we show the way we derive the exercise boundaries. The holder's boundary is the largest x such that $\alpha(x) = 0$. It is presented by the red point in Figure 1a—its value is 5.8562. The writer's boundary is the smallest x for which $\beta(x) = 0$—the red point in Figure 1b—the value is 17.8275. The results are based on the three-step algorithm with parameters $\tau = 3$, $r = -0.03$, $\lambda = 0.05$, $\sigma = 0.3$, $k = 20$, and $\eta = 1$.

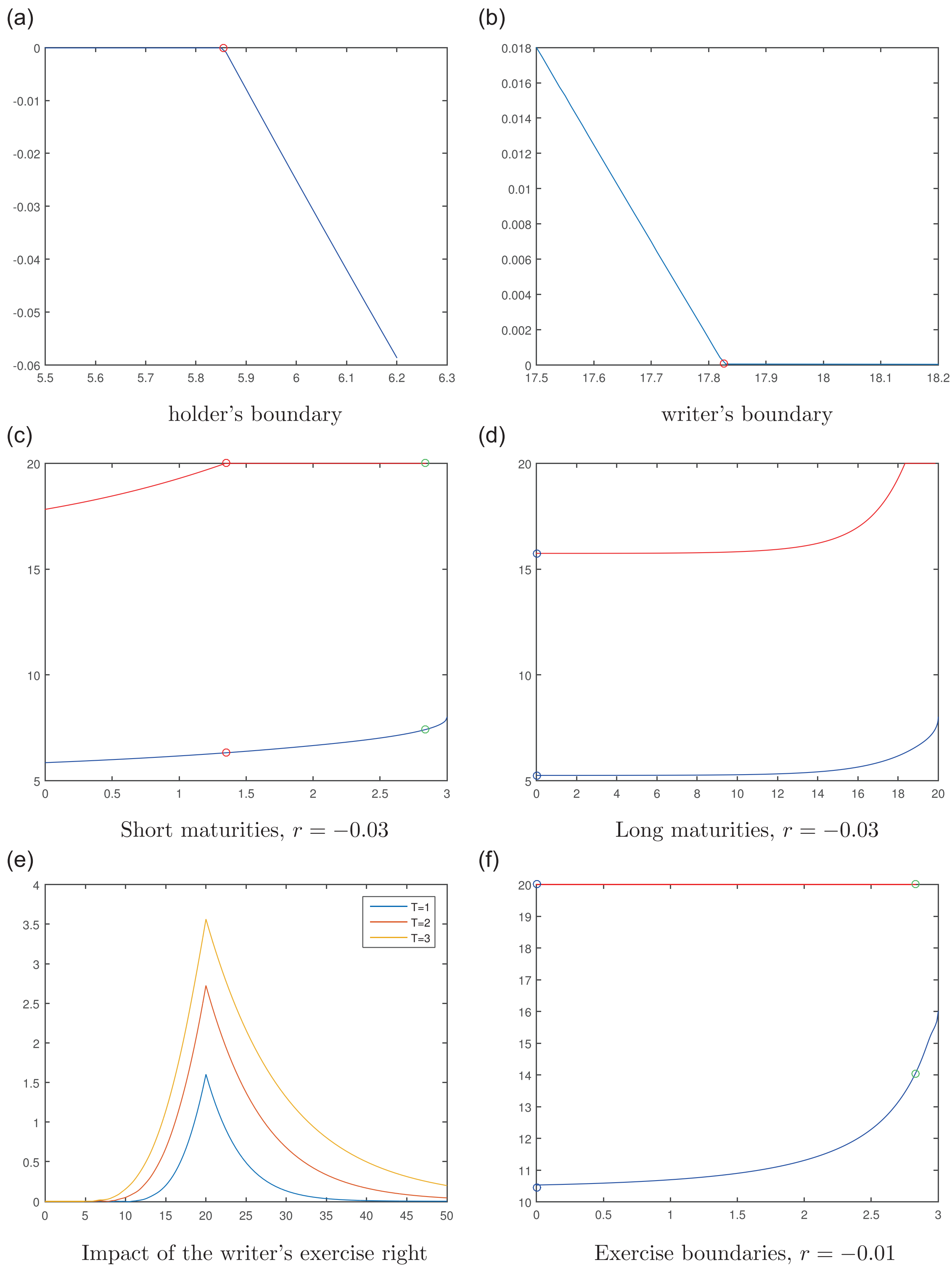

FIGURE 1 Exercise boundaries: (a,b) deriving the boundaries—the red points are our approximations; (c–f) both boundaries with different parameters—the red upper line is the writer's boundary; the blue line is the holder's one [Color figure can be viewed at wileyonlinelibrary.com]

5 | PRICING

If the time to maturity is less than the first critical value τ_1, the option turns to an ordinary American put. Suppose now $T \equiv \tau > \tau_1$. We have proved that the continuation region consists of two parts—one between the optimal boundaries, and one above the strike. We shall examine these cases separately.

5.1 | Initial asset value between the optimal boundaries

We have to evaluate price function (21) knowing the already approximated boundaries. We shall create an efficient and relatively fast Monte Carlo algorithm to derive numerically the expectations in function (21). It is based on the simulations of the Brownian motion's sample paths and a numerical evaluation of the integrals in expectations (A12)–(A14), provided that the Brownian motion stays in the strip. The algorithm is as follows.

1. We generate randomly $n-1$ numbers using the standard normal distribution. These numbers form the vector $\bar{u}$.
2. Let $m \leq n$ and the vector u consists of the first $m-1$ elements of $\bar{u}$. Let D be an $(m-1)\times(m-1)$ diagonal matrix with elements $\sqrt{\Delta t_i/n}$. Note that the length of the intervals, Δt_i, differs before and after the moment $T-\tau_1$. We calculate the vector x as $x = MDu$, where M is an $(m-1)\times(m-1)$ lower triangle matrix with value one.
3. If $t_{m-1} < T - \tau_1$ we derive the values v as
$$v = v(x_1, \ldots, x_{m-1}) = \prod_{i=1}^{m-1} I_{c_i<x_i<d_i}\left(1 - \sum_{j=1}^{\infty} q_{ij}(x_{i-1}, x_i)\right). \tag{30}$$
Otherwise, if $p \leq m-1$, then v is obtained as
$$\begin{aligned} v &= v(x_1, \ldots, x_{m-1}) = \prod_{i=1}^{p-1} I_{c_i<x_i<d_i}\left(1 - \sum_{j=1}^{\infty} q_{ij}(x_{i-1}, x_i)\right) \\ &\times \prod_{i=p}^{m-1} I_{c_i<x_i}\left(1 - \exp\left(-\frac{2(c_{i-1}-x_{i-1})(c_i - x_i)}{\Delta t_i}\right)\right). \end{aligned} \tag{31}$$
4. We derive the values w as $w = e^{-\xi t_{m-1}} L_{1,2}(\cdot)$ for Equations (A12) and (A13) and $w = e^{\xi x_{n-1}} V(\cdot)$ for Equation (A14).
5. We calculate the product $P = vw$.
6. We repeat the procedure above H times and after averaging we derive the necessary expectations as $\frac{1}{H}\sum_{i=1}^{H} P_i$.

Let us give some comments on the algorithm above. In steps 1 and 2 we simulate the Brownian motion paths. The term v from step 3 pertains to the requirement that the Brownian motion stays in the strip till the moment t_{m-1}. If $t_{m-1} < T - \tau_1$ we have to use the form (A12) of Proposition A.7. Otherwise, if $t_{m-1} \geq T - \tau_1$, the writer's exercise boundary does not exist after some moment and therefore the strip is open above. In this case we have to use Equation (A13). It turns out that five iterations are sufficient in the infinite sum except for some extremely large parameters' values when 10 iterations are necessary. The term w in step 4 has two meanings. If the first exit is before the maturity, w is related to the first exit in the interval $(t_{m-1}, t_m]$. We use L_1 if the exit happens from the lower (holder's) boundary, whereas L_2 is used for the upper (writer's) boundary. Otherwise, if the Brownian motion stays in the strip till the maturity, then w pertains to the terminal option payment at the maturity. The fifth term in option price formula (21) is obtained for $\xi = 0$ in Equation (A14). The last two steps allow us to derive the corresponding expectations. Note that the terms related to the transition density of the Brownian motion in Equations (A12)–(A14) are incorporated hiddenly when we generate the Brownian motion's sample paths. For some more comments see Wang and Pötzelberger (1997) and Pötzelberger and Wang (2001).

5.2 | Initial asset value above the strike

Suppose now that $S_0 \equiv x > K$—pricing turns into a one-sided hitting problem. The option expires when the underlying asset hits the strike if the remaining time to maturity is more than τ_1 and pays an amount of η. If the underlying asset stays above the strike until the moment $\tau - \tau_1$, then the option turns to a noncancellable American one. Let us denote by ζ the first hitting moment, and by d, a_1, a_2, and $f(y; t)$, the following terms:

$$\begin{aligned} a_1 &= -\frac{r}{\sigma} + \frac{\sigma}{2}, \\ a_2 &= -\frac{\ln S_0 - \ln K}{\sigma}, \\ d &= a_1(T - \tau_1) + a_2, \\ f(y;\, t) &= \frac{1}{\sqrt{2\pi t}}\left(1 - \exp\left(-\frac{2a_2(a_1 t + a_2 - y)}{t}\right)\right)\exp\left(-\frac{y^2}{2t}\right). \end{aligned} \tag{32}$$

Using Propositions A.2 and A.3 we derive the following semiclosed formula:

$$\begin{aligned} V(\tau, S_0) &= E\left[e^{-(r+\lambda)\zeta}\eta I_{\zeta \le T-\tau_1}\right] \\ &\quad + e^{-r(T-\tau_1)}\int_d^\infty e^{-\lambda(T-\tau_1)}V_{\mathrm{am}}\left(\tau_1,\, S_0 e^{\left(r-\frac{\sigma^2}{2}\right)(T-\tau_1)+\sigma y}\right)dQ\left(B_{T-\tau_1} < y, \zeta > T-\tau_1\right) \\ &= \eta e^{-a_2\left(\sqrt{a_1^2+2(r+\lambda)}+a_1\right)}g\left(T-\tau_1, -\sqrt{a_1^2 + 2(r+\lambda)}, a_2\right) \\ &\quad + e^{-(r+\lambda)(T-\tau_1)}\int_d^\infty V_{\mathrm{am}}\left(\tau_1,\, S_0 e^{\left(r-\frac{\sigma^2}{2}\right)(T-\tau_1)+\sigma y}\right)f(y;\, T-\tau_1)\,dy, \end{aligned} \tag{33}$$

where the function $g(\cdot,\cdot,\cdot)$ is given in Equation (A4).

6 | NUMERICAL RESULTS

We present the results of some numerical experiments in this section. The main values we use are—time to maturity $\tau = 3$; risk-free rate $r = -0.03$; discount rate $\lambda = 0.05$; volatility $\sigma = 0.3$; strike $K = \$20$; penalty $\eta = \$1$. We shall vary some of them to describe the option's behavior. The Brownian motion's paths are simulated by n =200,000 steps. We divide the time interval into 16 subintervals. For each node of this grid we use a procedure with three steps to derive the boundaries' approximations.

First, by the use of Zaevski (2020b) we derive the exercise boundaries for the related perpetual option. We have that the writer's optimal boundary is $B = 15.7208$ (the holder's one is $A = 5.2520$). Since it is less than the strike we have to use the presented in Section 4.2 algorithm. We derive the first critical value, τ_1, at which the pure American option price is equal to the penalty. It turns out that it is $\tau_1 = 0.1604$. Hence, the option is noncancellable when the time to maturity is less than τ_1. So, the writer's boundary does not exist for $\tau \in [0, 0.1604]$ and therefore the holder's boundary can be obtained using Zaevski (2021b). The given in Equation (15) holder's boundary value at the maturity is currently \$8. It turns out that the second important time to maturity value at which the writer's boundary falls below the strike is $\tau_2 = 1.3459$. Both boundaries for short maturities are presented in Figure 1c. Note that the time dependence is presented by the actual time, not by the time to maturity. The writer's boundary is the upper line, which is plotted by red color. The holder's boundary is the lower blue line. By red and green are marked the critical values τ_1 and τ_2. The long maturity behavior is presented in Figure 1. The blue points are the corresponding perpetual values. We can see that the long maturity values tend to the perpetual ones.

The boundaries' surface w.r.t. the discount rate, $\lambda \in [0.04, 0.1]$, and for short maturities, $\tau \in [0, 3]$, can be viewed in Figure 2a,c, whereas the long maturities behavior, $\tau \in [3, 20]$, is presented in Figure 2b,d. Note that here the time dependence is w.r.t. the time to maturity. Once we approximate the boundaries, we can use the presented in Section 5 Monte Carlo method to evaluate the options assuming that the initial asset price is $S_0 = \$15$. The price behavior for short and long maturities can be viewed in Figure 2e,f. The meaning of the green, red, and blue points is preserved. We can see that the long maturity values tend to the perpetual ones. Some particular prices are presented in Table 1. The discount rate is among $\lambda \in \{0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1\}$, the penalty varies as $\eta \in \{\$0.1, \$1, \$2, \$3, \$4, \$5\}$, the initial asset value is taken to be $x = \$10$, $x = \$11$, $x = \$12$, $x = \$13$, or $x = \$14$.

In Figure 1e we compare the cancellable and noncancellable options presenting the difference between their prices. In such a way we can appreciate the impact of the writer's early cancelling right. The initial asset price is varied in the

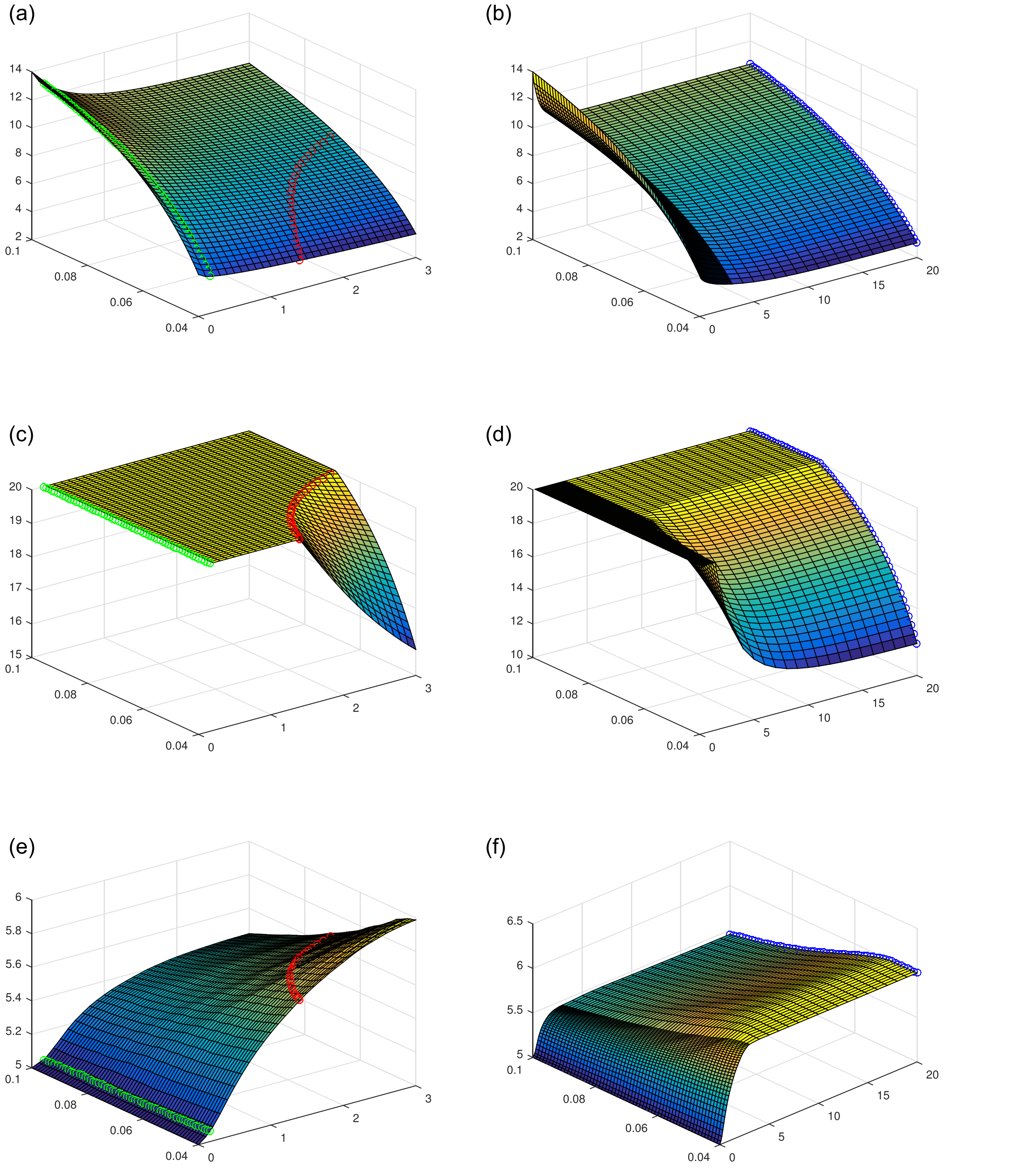

FIGURE 2 Exercise boundaries and put option prices: (a) holder's boundary—short maturities, (b) holder's boundary—long maturities, (c) writer's boundary—short maturities, (d) writer's boundary—long maturities, (e) prices—short maturities, and (f) prices—long maturities [Color figure can be viewed at wileyonlinelibrary.com]

interval $(0, 50)$. The time to maturity is assumed to be $T = 1$, $T = 2$, or $T = 3$. The holder's boundaries at these moments are 6.6643, 6.1739, and 5.8559, whereas the writer's ones are 20, 19.2911, and 17.8291. We can see that when $T = 1$ we have a (τ_1, η)-American option, whereas for $T = 2$ and $T = 3$ the option is a real cancellable one (note that the second critical value has been obtained as $\tau_2 = 1.3459$). We can see that the prices of both options tend one to other when $S_0 \to 0$ or $S_0 \to \infty$. When $S_0 \to 0$ the limit is the strike, because it is very likely S_t to stay near the zero. Analogously, when the option is deeply out-of-the-money, then $S_t > K$ with a very high probability and therefore both

TABLE 1 Option prices

$S_0 = 10$	$\eta = 0.1$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$	$\eta = 5$
$S_0 = 10$						
$\lambda = 0.04$	10.1000	10.5802	10.6633	10.7234	10.7776	10.8319
$\lambda = 0.05$	10.0919	10.3330	10.4129	10.4720	10.5255	10.5791
$\lambda = 0.06$	10.0392	10.1818	10.2547	10.3107	10.3628	10.4149
$\lambda = 0.07$	10.0022	10.0772	10.1422	10.1963	10.2468	10.2969
$\lambda = 0.08$	10.0000	10.0201	10.0648	10.1100	10.1551	10.2008
$\lambda = 0.09$	10.0000	10.0011	10.0225	10.0540	10.0904	10.1299
$\lambda = 0.1$	10.0000	10.0000	10.0032	10.0207	10.0470	10.0786
$S_0 = 11$						
$\lambda = 0.04$	9.1000	9.6897	9.8207	9.9196	10.0112	10.1028
$\lambda = 0.05$	9.1000	9.4571	9.5813	9.6791	9.7696	9.8600
$\lambda = 0.06$	9.0745	9.2942	9.4129	9.5076	9.5962	9.6850
$\lambda = 0.07$	9.0251	9.1797	9.2889	9.3794	9.4649	9.5513
$\lambda = 0.08$	9.0022	9.0953	9.1932	9.2802	9.3629	9.4466
$\lambda = 0.09$	9.0000	9.0407	9.1200	9.1976	9.2752	9.3553
$\lambda = 0.1$	9.0000	9.0123	9.0670	9.1331	9.2039	9.2772
$S_0 = 12$						
$\lambda = 0.04$	8.1000	8.7882	8.9794	9.1305	9.2727	9.4150
$\lambda = 0.05$	8.1000	8.5779	8.7590	8.9080	9.0486	9.1893
$\lambda = 0.06$	8.0931	8.4129	8.5878	8.7341	8.8730	9.0119
$\lambda = 0.07$	8.0600	8.2946	8.4610	8.6039	8.7398	8.8758
$\lambda = 0.08$	8.0244	8.2026	8.3575	8.4938	8.6260	8.7590
$\lambda = 0.09$	8.0047	8.1300	8.2732	8.4043	8.5317	8.6599
$\lambda = 0.1$	8.0000	8.0777	8.2019	8.3256	8.4490	8.5738
$S_0 = 13$						
$\lambda = 0.04$	7.1000	7.8711	8.1405	8.3575	8.5648	8.7721
$\lambda = 0.05$	7.1000	7.6968	7.9465	8.1605	8.3656	8.5707
$\lambda = 0.06$	7.1000	7.5370	7.7815	7.9928	8.1957	8.3988
$\lambda = 0.07$	7.0858	7.4167	7.6497	7.8568	8.0568	8.2571
$\lambda = 0.08$	7.0576	7.3222	7.5470	7.7493	7.9455	8.1420
$\lambda = 0.09$	7.0302	7.2466	7.4578	7.6539	7.8464	8.0396
$\lambda = 0.1$	7.0119	7.1820	7.3825	7.5700	7.7563	7.9449
$S_0 = 14$						
$\lambda = 0.04$	6.1000	6.9388	7.3059	7.6017	7.8879	8.1742
$\lambda = 0.05$	6.1000	6.7984	7.1308	7.4234	7.7070	7.9905
$\lambda = 0.06$	6.1000	6.6584	6.9799	7.2693	7.5503	7.8313
$\lambda = 0.07$	6.0992	6.5399	6.8549	7.1402	7.4182	7.6964
$\lambda = 0.08$	6.0850	6.4485	6.7510	7.0313	7.3057	7.5805
$\lambda = 0.09$	6.0630	6.3720	6.6659	6.9395	7.2091	7.4798
$\lambda = 0.1$	6.0414	6.3088	6.5889	6.8574	7.1229	7.3894

prices tend to zero. Obviously, we have not a smoothness in the strike. We can see also that the impact of the writer's cancelling right is more significant when the underlying asset starts near the strike. Also, this impact is larger for the higher maturities.

If the risk-free rate is $r = -0.01$ then the perpetual writer's boundary is $B = \$20$. Since it is equal to the strike, the option turns to (τ_1, η)-American. Hence, we have to use the algorithm presented in Section 4.1. The first critical value is $\tau_1 = 0.1719$. Of course, the second one is the infinity. Both boundaries are presented in Figure 1f.

7 | CONCLUSIONS

The pricing problem for a cancellable American put option written on a dividend-paying asset has been examined in this paper. This is done through an additional discount factor. The maturity is assumed to be finite. A series of propositions that characterize the optimal regions are proven. Both early exercise boundaries are approximated numerically maximizing the financial utilities of both option's participants. A Monte Carlo method for option pricing has been constructed when the asset starts between the exercise boundary. Alternatively, a semiclosed pricing formula has been derived if the initial asset price is above the strike. We have validated the consistency and relevance of the derived results by performing various numerical experiments.

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DATA AVAILABILITY STATEMENT

No data are created or used during the work on this paper.

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APPENDIX A: FIRST HITTING TIME PROPOSITIONS

Let

$$\begin{aligned} a(t) &= (a_{1,m}t + a_{2,m})I_{t_{m-1}<t\le t_m}, \\ b(t) &= (b_{1,m}t + b_{2,m})I_{t_{m-1}<t\le t_m} \end{aligned} \tag{A1}$$

be two continuous piecewise linear functions w.r.t. the intervals $0 \equiv t_0 < t_1 < \cdots < t_n \equiv T$. We impose the conditions $a(0) < 0 < b(0)$, and $a(t) < b(t)$ for all $t \le T$. Let the upper boundary vanishes after some moment t_p, $p \le n$. We can think that $b_{1,m} = b_{2,m} = \infty$ for $p \le m \le n$. Let the corresponding values at the endpoints be $\alpha_i = a(t_i)$ and $\beta_i = b(t_i)$ and γ_i be the difference between them, $\gamma_i = \beta_i - \alpha_i$. Note that $\gamma_i > 0$. We shall denote by τ_1 and τ_2 the first hitting times of a Brownian motion to the functions $a(t)$ and $b(t)$, respectively, and by τ the lower one, $\tau = \tau_1 \wedge \tau_2$. Let $N(x)$ be the cumulative distribution function of the standard normal distribution. We shall use the following notations:

$$\begin{aligned} q_{i,j}(y,z) = \exp&\left(-\frac{2[j\gamma_{i-1} + \alpha_{i-1} - y][j\gamma_i + \alpha_i - z]}{t_i - t_{i-1}}\right) \\ - \exp&\left(-\frac{2j[j\gamma_{i-1}\gamma_i + \gamma_{i-1}(\alpha_i - z) - \gamma_i(\alpha_{i-1} - y)]}{t_i - t_{i-1}}\right) \\ + \exp&\left(-\frac{2[j\gamma_{i-1} - (\beta_{i-1} - y)][j\gamma_i - (\beta_i - z)]}{t_i - t_{i-1}}\right) \\ - \exp&\left(-\frac{2j[j\gamma_{i-1}\gamma_i - \gamma_{i-1}(\beta_i - z) + \gamma_i(\beta_{i-1} - y)]}{t_i - t_{i-1}}\right) \end{aligned} \tag{A2}$$

for $\alpha_{i-1} < y < \beta_{i-1}$ and $\alpha_i < z < \beta_i$. Note that if $i \ge p$, then

$$\begin{aligned} q_{i,1}(y,z) &= \exp\left(-\frac{2(\alpha_{i-1} - y)(\alpha_i - z)}{t_i - t_{i-1}}\right) \\ q_{i,j}(y,z) &= 0, \quad j > 1. \end{aligned} \tag{A3}$$

Proposition A.1. *The probability that the Brownian motion hits the linear function $a(t) = a_1 t + a_2$, $a_2 < 0$, till moment T is given by the equation*

$$g(T; a_1, a_2) \equiv P(\tau_1 < T) = N\left(\frac{a_1 T + a_2}{\sqrt{T}}\right) + \exp(-2a_1 a_2)N\left(\frac{-a_1 T + a_2}{\sqrt{T}}\right). \tag{A4}$$

Proof. The proof can be found in Zaevski (2020c), proposition 3.1. □

Proposition A.2. *Let $\xi > 0$ and the function $a(t) = a_1 t + a_2$, $a_2 < 0$, be linear. Then the truncated Laplace transform of the first hitting time of the Brownian motion to it is given by the equation*

$$L(T, \xi; a_1, a_2) = E[e^{-\xi\tau_1} I_{\tau \le T}] = e^{-a_2\left(\sqrt{a_1^2 + 2\xi} + a_1\right)} g\left(T; -\sqrt{a_1^2 + 2\xi}, a_2\right), \tag{A5}$$

where $g(\cdot)$ is the function given by Equation (A4).

Proof. The proposition is proven in Zaevski (2020c), theorem 3.1. □

Proposition A.3. *If $y > a(t)$, then*

$$P(B_t < y, \tau > t) = \int_{-\infty}^{y} f(u; t)\, du, \tag{A6}$$

where the function $f(\cdot;\cdot)$ is given in Equation (32).

Proof. See eq. (3.7) from Zaevski (2020c). □

Proposition A.4. *If the first hitting time of a Brownian motion to the linear function $a(t) = a_1 t + a_2$, $a_2 < 0$, is after the moment T, then for $z > a(T)$*

$$\begin{aligned} V(\xi, z, T; a_1, a_2) &\equiv E\left[e^{\xi B_T} I_{B_T < z, \tau_1 > T}\right] \\ &= \exp\left(\frac{T\xi^2}{2}\right) \left[\begin{array}{l} N\left(\frac{z - T\xi}{\sqrt{T}}\right) - N\left(\frac{a(T) - T\xi}{\sqrt{T}}\right) \\ - e^{2a_2(\xi - a_1)}\left(N\left(\frac{z - T\xi - 2a_2}{\sqrt{T}}\right) - N\left(\frac{a(T) - T\xi - 2a_2}{\sqrt{T}}\right)\right) \end{array}\right]. \end{aligned} \tag{A7}$$

Proof. The proof can be found in Zaevski (2020c), theorem 3.2. □

Proposition A.5. *Let the functions $a(\cdot)$ and $b(\cdot)$ be linear. Then the probabilities the first exit from the strip to be before the moment T are given by*

$$\begin{aligned} P_1^l(T; a_1, a_2, b_1, b_2) &\equiv P((\tau_1 \wedge \tau_2) < T, \tau_1 < \tau_2) \\ &= N\left(\frac{a_1 T + a_2}{\sqrt{T}}\right) \\ &+ \sum_{j=1}^{\infty} \left\{\begin{array}{l} e^{-2[-ja_2 + (j-1)b_2][-ja_1 + (j-1)b_1]} N\left(\frac{-a_1 T - 2(j-1)b_2 + (2j-1)a_2}{\sqrt{T}}\right) \\ - e^{-2[j^2(a_1a_2 + b_1b_2) - j(j-1)a_2b_1 - j(j+1)b_2a_1]} N\left(\frac{-a_1 T - 2jb_2 + (2j-1)a_2}{\sqrt{T}}\right) \\ - e^{-2[-(j-1)a_2 + jb_2][-(j-1)a_1 + jb_1]} N\left(\frac{a_1 T - 2jb_2 + (2j-1)a_2}{\sqrt{T}}\right) \\ + e^{-2[j^2(a_1a_2 + b_1b_2) - j(j-1)b_2a_1 - j(j+1)a_2b_1]} N\left(\frac{a_1 T + (2j+1)a_2 - 2jb_2}{\sqrt{T}}\right) \end{array}\right\} \end{aligned} \tag{A8}$$

and

$$
\begin{aligned}
P_2^l(T; a_1, a_2, b_1, b_2) &\equiv P((\tau_1 \wedge \tau_2) < T, \tau_2 < \tau_1) \\
&= 1 - N\left(\frac{b_1 T + b_2}{\sqrt{T}}\right) \\
&+ \sum_{j=1}^{\infty} \left\{ \begin{array}{l}
e^{-2[jb_2-(j-1)a_2][jb_1-(j-1)a_1]} N\left(\frac{b_1 T + 2(j-1)a_2 - (2j-1)b_2}{\sqrt{T}}\right) \\
- e^{-2[j^2(b_1b_2+a_1a_2)-j(j-1)b_2a_1-j(j+1)a_2b_1]} N\left(\frac{b_1 T + 2ja_2 - (2j-1)b_2}{\sqrt{T}}\right) \\
- e^{-2[(j-1)b_2-ja_2][(j-1)b_1-ja_1]} N\left(\frac{-b_1 T + 2ja_2 - (2j-1)b_2}{\sqrt{T}}\right) \\
+ e^{-2[j^2(b_1b_2+a_1a_2)-j(j-1)a_2b_1-j(j+1)b_2a_1]} N\left(\frac{-b_1 T - (2j+1)b_2 + 2ja_2}{\sqrt{T}}\right)
\end{array} \right\}.
\end{aligned} \tag{A9}
$$

Proof. The proof can be found in theorem 4.3 from Anderson (1960). □

Proposition A.6. *Let $\xi > 0$. Then the truncated Laplace transforms of the first hitting times of a Brownian motion to the linear functions $a(\cdot)$ and $b(\cdot)$ are*

$$
\begin{aligned}
L_1(t, \xi; a_1, a_2, b_1, b_2) &= E\left[e^{-\xi\tau} I_{\tau \le T, \tau = \tau_1}\right] \\
&= e^{a_2\left(\sqrt{a_1^2+2\xi}-a_1\right)} P_1^l\left(T; \sqrt{a_1^2 + 2\xi}, a_2, b_1 + \sqrt{a_1^2 + 2\xi} - a_1, b_2\right),
\end{aligned} \tag{A10}
$$

$$
\begin{aligned}
L_2(t, \xi; a_1, a_2, b_1, b_2) &= E\left[e^{-\xi\tau} I_{\tau \le T, \tau = \tau_2}\right] \\
&= e^{b_2\left(\sqrt{b_1^2+2\xi}-b_1\right)} P_2^l\left(T; a_1 + \sqrt{b_1^2 + 2\xi} - b_1, a_2, \sqrt{b_1^2 + 2\xi}, b_2\right),
\end{aligned} \tag{A11}
$$

where the functions $P_1^l(\cdot)$ and $P_2^l(\cdot)$ are given by Equations (A8) and (A9), respectively. Note that the first boundary stays below the second one after changing the coefficients.

Proof. The proof can be found in Zaevski (2021a), theorem 3.1. □

Proposition A.7. *The truncated Laplace transforms of the first hitting times of a Brownian motion to the piecewise linear functions (A1) are given by the equations*

$$
\begin{aligned}
&E\left[e^{-\xi\tau} I_{\tau \in (t_{m-1}, t_m)}, \tau = \tau_{1,2}\right] \\
&= \int_{\alpha_1, \ldots, \alpha_{m-1}}^{\beta_1, \ldots, \beta_{m-1}} \left(\begin{array}{l}
\prod_{i=1}^{m-1} \left(1 - \sum_{j=1}^{\infty} q_{i,j}(x_{i-1}, x_i)\right) \dfrac{\exp\left(-\dfrac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)}{\sqrt{2\pi(t_i - t_{i-1})}} \\
e^{-\xi t_{m-1}} L_{1,2}(t_m - t_{m-1}, \xi; a_{m,1}, \alpha_{m-1} - x_{m-1}, b_{m,1}, \beta_{m-1} - x_{m-1})
\end{array} \right) dx_1 \cdots dx_{m-1},
\end{aligned} \tag{A12}
$$

where the functions $L_1(\cdot)$ and $L_2(\cdot)$ are given by Equations (A10) and (A11). If $m > p$, then Equation (A3) holds and thus formula (A12) turns to

$$E\left[e^{-\xi\tau}I_{\tau\in(t_{m-1},t_m)},\tau=\tau_1\right]$$
$$=\int_{\alpha_1,\ldots,\alpha_{m-1}}^{\beta_1,\ldots,\beta_{k-1},\infty}\left(\begin{array}{l}\prod_{i=1}^{p-1}\left(1-\sum_{j=1}^{\infty}q_{i,j}(x_{i-1},x_i)\right)\frac{\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)}{\sqrt{2\pi(t_i-t_{i-1})}}\\ \prod_{i=p}^{m-1}\left(1-\exp\left(-\frac{2(\alpha_{i-1}-x_{i-1})(\alpha_i-x_i)}{t_i-t_{i-1}}\right)\right)\frac{\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)}{\sqrt{2\pi(t_i-t_{i-1})}}\\ e^{-\xi t_{m-1}}L(t_m-t_{m-1},\xi;a_{m,1},\alpha_{m-1}-x_{m-1})\end{array}\right)dx_1\cdots dx_{m-1}, \tag{A13}$$

where $L(\cdot)$ is the function from Equation (A5).

Proof. The proof can be found in Zaevski (2021a), theorems 4.1 and 4.2. □

Proposition A.8. *The Laplace transform of the Brownian motion if τ is after T is*

$$E\left[e^{\xi B_T}I_{B_T<z,\tau>T}\right]$$
$$=\int_{\alpha_1,\ldots,\alpha_{n-1}}^{\beta_1,\ldots,\beta_{k-1},\infty}\left(\begin{array}{l}\prod_{i=1}^{p-1}\left(1-\sum_{j=1}^{\infty}q_j(x_{i-1},x_i;i)\right)\frac{\left(\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)\right)}{\sqrt{2\pi(t_i-t_{i-1})}}\\ \prod_{i=p}^{n-1}\left(1-\exp\left(-\frac{2(\beta_{i-1}-x_{i-1})(\beta_i-x_i)}{t_i-t_{i-1}}\right)\right)\frac{\left(\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)\right)}{\sqrt{2\pi(t_i-t_{i-1})}}\\ e^{\xi x_{n-1}}V(\xi,z-x_{n-1},t_n-t_{n-1};a_{1,n-1},\alpha_{n-1}-x_{n-1})\end{array}\right)dx_1\cdots dx_{n-1}. \tag{A14}$$

The function $V(\cdot)$ is given by formula (A7).

Proof. The proof can be found in Zaevski (2021a), theorem 4.2. □